Even Abundant Numbers.

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1. Theorem. *There is only a finite number of primitive non-deficient numbers having a given number n of distinct odd prime factors and a given number m of factors 2.*

Let $p_1, \ldots, p_n$ be primes in ascending order. Then

$$a = 2^m p_1^{e_1} p_2^{e_2} \ldots p_n^{e_n}$$

is deficient if

$$G \cdot \left(\frac{p_1}{p_1 - 1}\right)^n < 2, \quad G \equiv \frac{2^{m+1}-1}{2^m},$$

and hence if $p_1 > r/(r-1)$, where r is the positive real n-th root of $2/G$, whence $r > 1$. Thus the least odd prime factor p_1 of a non-deficient number is limited. To prove by induction a like result for each p_i, let $p_1, \ldots, p_\nu$ be given primes, and $\nu < n$. The divisor

$$\alpha = 2^m p_1^{e_1} \ldots p_\nu^{e_\nu}$$

of a primitive a must be deficient. By Lemma B of the former paper, each deficient α is a product of some one of a finite number of α's by a number having no prime factor other than $p_1, \ldots, p_\nu$. We may now complete the proof as in the former paper, inserting in Π_μ the additional factor G.

For example, the primitive non-deficient $2^m p^e$ are $2^m p$, where $2^m - 1 < p \leqq 2^{m+1} - 1$.

2. In the determination of all even non-deficient numbers not exceeding a given number L, use may be made of the following theorems.

Theorem. *Let $2^l \leqq L < 2^{l+1}$ and $\lambda = l/2$ or $(l+1)/2$, according as l is even or odd. If n is any integer for which $l > n \geqq \lambda$ and k is any odd integer for which $2^n k \leqq L$, then $2^n k$ is non-deficient.*

We have $2^\lambda k \leqq 2^n k \leqq L < 2^{l+1}$. Now $l + 1 = 2\lambda + 1$ or 2λ, according as l is even or odd. Thus $k < 2^{n+1}$ or 2^n, in the respective cases. In each case, $k \leqq N$, where $N = 2^{n+1} - 1$. Thus k has a prime factor $p \leqq N$. Since $2^n p$ is non-deficient, the same is true of its multiple $2^n k$.

3. Theorem. *Let ν be the least integer for which*

$$2^\nu (2^{\nu+1} - 1)^2 \geqq L,$$

and n any integer $\geqq \nu$. Let c be any odd composite number for which $2^n c < L$. Then $2^n c$ is abundant.

Set $N = 2^{n+1} - 1$. Then $2^n N^2 \geqq L$ and $c < N^2$. Hence the composite number c has a prime factor $p < N$. Since $2^n p$ is abundant, its multiple $2^n c$ is abundant.

This theorem, in connection with the fact that $2^n p$ (p an odd prime) is abundant if and only if $p < N$, enables one to write down immediately all abundant numbers $2^n k < L$, where $n \geqq \nu$ (cf. § 8).

4. THEOREM. *Set $s = 2^{\nu+1} - 1$, $t = 2^\nu - 1$. The non-deficient even numbers $2^{\nu-1} k$ less than* $2^\nu s^2$ are those in which k has a prime factor $\leqq t$ and those in which k is a product of two primes $t + P$ and $t + Q$, where $0 < P < t$, $P < Q \leqq (t^2 + t)/P$. If $P = t - 1$, then $Q = t + 1$.*

Let every prime factor of k exceed t. Since

$$(2^\nu + 1)^3 \geqq 2 s^2$$

for every $\nu > 0$, k can not have three prime factors. If p is an odd prime, $2^n p^e$ is deficient if

$$\frac{2^{n+1} - 1}{2^n} \frac{p}{p - 1} < 2, \quad p > 2^{n+1}$$

Hence $2^{\nu-1} p^e$ is deficient if $p > t$. Finally, if p, q are distinct primes $> t$ and $q > p$, then $2^{\nu-1} p q$ is non-deficient if and only if

$$p q \leqq t (p + q + 1).$$

Set $p = t + P$, $q = t + Q$. Then the condition becomes $P Q \leqq t^2 + t$. But $Q \geqq P + 2$. Hence $P < t$. If $P = t - 1$, then $Q < t + 3$, so that $Q = t + 1$. For, if $Q \geqq t + 3$,

$$P Q \geqq t^2 + t + t - 3, \quad P Q > t^2 + t.$$

Indeed, $\nu \geqq 2$, $t \geqq 3$; while if $t = 3$ and if $Q = t + 3$, then $q = 9$.

5. THEOREM. *The non-deficient numbers $2^{\nu-2} k$ less than $2^\nu s^2$ and having† $\nu > 3$ are those in which k has a prime factor $\leqq r$, where $r = 2^{\nu-1} - 1$, those in which k is the product of two primes $r + P$ and $r + Q$, where $0 < P < r$, $P < Q \leqq (r^2 + r)/P$, and the following abundant numbers:*

$2^4 37 \cdot 41^2$, $2^4 37^2 q (q = 41, 43, 47)$, $2^3 \cdot 17 \cdot 19 l (23 \leqq l \leqq 47)$,
$2^3 17 \cdot 23 l (l = 29, 31, 37)$, $2^3 17 \cdot 29 \cdot 31$, $2^3 19 \cdot 23 l (l = 29, 31)$,
$2^3 17^2 q (q < 55)$, $2^3 19^2 q (q < 45)$, $2^3 23^2 q (q < 31)$, $2^3 29^2 17$,
$2^2 11^2 q (q < 26)$, $2^2 13^2 q (q < 19)$, $2^2 17^2 q (q < 14)$,
$2^2 11 \cdot 13 l (l = 17, 19, 23)$, $2^2 11 \cdot 17 \cdot 19$,

where q and l are primes, $q > 7$ in the fourth line, $q > 15$ in the third line.

Let every prime factor of k exceed r. Since

$$(2^{\nu-1} + 1)^4 > 2^{4\nu-4} \geqq 4 \cdot 2^{2\nu+2} > 4 s^2$$

* Including all $< L$, since $L \leqq 2^\nu s^2$ by § 3.
† Those with $\nu = 3$ are given in § 7.

for $\nu > 3$, k can not have four prime factors. Since

$$(2^{\nu-1}+1)^3 > 2^{3\nu-3} \geqq 4 \cdot 2^{2\nu+2} > 4\,s^2$$

for $\nu \geqq 7$, k has at most two prime factors unless $\nu = 6, 5$ or 4. As in § 4, $2^{\nu-2}p^e$ is deficient if $p > r$, since then $p > 2^{\nu-1}$. The case in which k is the product of two distinct prime factors may be treated as in § 4. We shall next consider $2^{\nu-2}p^2 q$, where $\nu = 6, 5$ or 4, and p, q are distinct primes $> r$. First, let $\nu = 6$, whence $r = 31$, $s = 127$. The least p is 37; by $37^2 q < 4s^2$, $q < 48$; $2^4 37^2 q$ is abundant if $q < 229$. For $p = 41$, $4s^2/p^2 < 39$, whence $q = 37$. Next, let $\nu = 5$, whence $r = 15$, $s = 63$. For $p = 17, 19, 23, 29, 31$, $p^2 q < 4s^2$ for $q < 55, 45, 31, 19, 17$, respectively, the final p being therefore excluded; while $2^3 p^2 q$ is abundant for $q < 243, 94, 50, 34, 31$, respectively. Finally, let $\nu = 4$, whence $r = 7$, $s = 31$. For $p = 11, 13, 17, 19$, $p^2 q < 4s^2$ for $q < 32, 23, 14, 11$; while $2^2 p^2 q$ is abundant for $q < 26, 19, 14, 13$.

It remains only to consider $2^{\nu-2}pql$, where p, q, l are distinct primes $> r$, arranged in ascending order. First, let $\nu = 6$. If $p \geqq 41$, $pql > 41^3 > 4\,s^2$. Hence $p = 37$. But $37 \cdot 41 \cdot 43 > 4\,s^2$. For $\nu = 5$ or 4, the numbers $< 4\,s^2$ are listed in the theorem, all being abundant.

6. While we might treat similarly the cases ν—3, etc., the results already obtained, together with those in § 7 for non-deficient numbers $2\,k$, enable us to tabulate in § 8 the even non-deficient numbers less than $2^\nu s^2$ for $\nu = 4$, namely, $< 2^4 31^2 = 15376$. Indeed, under this limit L, there is no primitive non-deficient number $2\,k$, where k is an odd number with more than three distinct prime factors (a case not treated in § 7). First, 3 is not a factor. If 5 is a factor, 7 is not, and $2\,k \geqq 2 \cdot 5 \cdot 11 \cdot 13 \cdot 17 = 24310 > L$. If 5 is not a factor, $2\,k \geqq 2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$, which exceeds the preceding.

7. THEOREM. *The primitive non-deficient numbers $2\,k$, where k is an odd number with at most three distinct prime factors, are:*

$2{\cdot}3$, $2{\cdot}5{\cdot}7$, $2{\cdot}5^2 11$, $2{\cdot}5^2 13$, $2{\cdot}5{\cdot}11\,p\,(13 \leqq p \leqq 53)$,
$2{\cdot}5{\cdot}11^2\,p\,(59 \leqq p \leqq 89$, $2{\cdot}5{\cdot}11^3 97$, $2{\cdot}5{\cdot}13\,p\,(17 \leqq p \leqq 31)$,
$2{\cdot}5{\cdot}13^2 37$, $2{\cdot}5{\cdot}17{\cdot}19$, $2{\cdot}5^2 17\,p\,(23 \leqq p \leqq 61)$, $2{\cdot}5^2 17^2\,p\,(67 \leqq p \leqq 79)$,
$2{\cdot}5^2 17^3 83$, $2{\cdot}5^3 17\,p\,(67 \leqq p \leqq 109)$, $2{\cdot}5^3 17^2 p\,(113 \leqq p \leqq 173)$,
$2{\cdot}5^3 17^3 179$, $2{\cdot}5^3 17^3 181^2$, $2{\cdot}5^3 17^4 181$, $2{\cdot}5^4 17\,p\,(p = 113, 127)$,
$2{\cdot}5^4 17^2\,p\,(179 \leqq p \leqq 223)$, $2{\cdot}5^4 17^3\,p\,(p = 227, 229, 233)$, $2{\cdot}5^5 17{\cdot}131$,
$2{\cdot}5^5 17^2\,p\,(p = 227, 229, 233)$, $2{\cdot}5^5 17^2 239^2$, $2{\cdot}5^5 17^3\,p\,(p = 239, 241)$,
$2{\cdot}5^5 17^3 251^2$, $2{\cdot}5^6 17^2\,p\,(p = 239, 241)$, $2{\cdot}5^6 17^3 251$,
$2{\cdot}5^2 19\,p\,(23 \leqq p \leqq 43)$, $2{\cdot}5^2 19{\cdot}47^2$, $2{\cdot}5^2 19^2\,p\,(p = 47, 53)$,
$2{\cdot}5^3 19\,p\,(47 \leqq p \leqq 61)$, $2{\cdot}5^3 19{\cdot}67^2$, $2{\cdot}5^3 19^2\,p\,(67 \leqq p \leqq 79)$,

$2\cdot5^3 19^3 83^2$, $2\cdot5^4 19\,p\,(p=67,71,73)$, $2\cdot5^4 19^2 p\,(p=83,89)$,
$2\cdot5^2 23\,p\,(p=29,31)$, $2\cdot5^3 23\,p\,(p=37,41)$, $2\cdot5^3 23^2 43$, $2\cdot5^4 23^2 47$,
$2\cdot5^3 29\cdot31^2$, $2\cdot5^3 29^2 31$, $2\cdot5^4 29\cdot31$,
$2\cdot7\cdot11\cdot13$, $2\cdot7^2 13^2 17$, $2\cdot7^3 13\cdot17^2$, $2.7^3 13^3 19^3$, $2\cdot7^3 13^4 19^2$, $2\cdot7^4 13^2 19^3$,
$2\cdot7^4 13^3 19^2$.

The proof is similar to that used in the former paper.

8. We are now in a position to give the even non-deficient numbers* $< L = 15000$. This L is just under the maximum limit given by $\nu = 4$ (§ 6). Moreover, this L was the convenient limit used in listing the primitive odd abundant numbers in the former paper.

The non-deficient numbers $2^n k < L$, with $n \geqq 4$, are (§§ 2, 3):
$2^{12}3$, $2^{11}k\,(k \leqq 7)$, $2^{10}k\,(k \leqq 13)$, $2^9 k\,(k \leqq 29)$, $2^8 k\,(k \leqq 57)$, $2^7 k\,(k \leqq 117)$,
$2^6 c\,(c \leqq 233)$, $2^6 p\,(p \leqq 127)$, $2^5 c\,(c \leqq 467)$, $2^5 p\,(p < 63)$, $2^4 c\,(c \leqq 937)$, $2^4 p\,(p \leqq 31)$,
where k, c, p are odd and > 1, c is composite and p prime.

The non-deficient $2^3 k < L$ are (§ 4) those with k having a prime factor $\leqq 13$ $(k \leqq 1875)$ and $k = 29\cdot31$, $23\,q\,(29 \leqq q \leqq 43)$, $19\,q\,(23 \leqq q \leqq 73)$, $17\,q\,(19 \leqq q \leqq 109)$. In the last case only the limit for abundance $(q \leqq 131)$ exceeded the limit required by L.

The non-deficient $2^2 k < L$ are (§ 5) those with $k < 3750$ having a factor 3, 5 or 7; those with $k = 11\,q\,(q = 13, 17, 19)$; and

$2^2\,11^2 q\,(q = 13, 17, 19, 23)$, $2^2\,13^2 q\,(q = 11, 17)$, $2^2\,17^2 q\,(q = 11, 13)$,
$2^2\,11\cdot13\,l\,(l = 17, 19, 23)$, $2^2\,11\cdot17\cdot19$.

The primitive non-deficient $2k < L$ are (§ 7) those in the first two lines of the theorem below.

The primitives are now found at once. For instance, $2^2 k$ with k having a factor 3, 5 or 7 is a multiple of one of the primitives $2\cdot3$, $2^2 5$, $2^2 7$. In the list of non-deficients $2^n k$, $n \geqq 4$, k or c is always less than the square of the maximum prime p giving a non-deficient $2^n p$. Hence we obtain the

THEOREM. *The 98 primitive even non-deficient numbers* < 15000 *are:*
$2\cdot3$, $2\cdot5\cdot7$, $2\cdot5^2 11$, $2\cdot5^2 13$, $2\cdot5\cdot11\,p\,(13 \leqq p \leqq 53)$,
$2\cdot5\cdot13\,p\,(17 \leqq p \leqq 31)$, $2\cdot5\cdot17\cdot19$, $2\cdot7\cdot11\cdot13$,
$2^2 5$, $2^2 7$, $2^2 11\,q\,(q = 13, 17, 19)$, $2^2 11^2 23$, $2^2 13\cdot17^2$, $2^2 13^2 17$,
$2^3 11$, $2^3 13$, $2^3 17\,q\,(19 \leqq q \leqq 109)$, $2^3 19\,q\,(23 \leqq q \leqq 73)$, $2^3 23\,q\,(29 \leqq q \leqq 43)$, $2^3 29\cdot31$,
$2^4 p\,(17 \leqq p \leqq 31)$, $2^5 p\,(37 \leqq p \leqq 61)$, $2^6 p\,(67 \leqq p \leqq 127)$.

* All < 6232 are tabulated in *Quar. Jour. Math.*, 1913, pp. 274–7.